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Spinning gas clouds with precession: non-degenerate cases

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Abstract
We extend here our earlier treatment (Gaffet B 2003 *J. Phys. A: Math. Gen.* **36** 5211, Gaffet B 2005 *Phil. Trans. R. Soc.* A accepted) of the evolution of spinning gas clouds with precession, to the non-degenerate cases where the last integral of motion L_6 is non-vanishing. The Liouville torus is still found to be transformable into a quartic surface, on which there are 15 conic point singularities. The differential system governing the evolution is integrable by quadratures: there is an integrating factor, and the independent variable u (which is the thermasy of the cloud) admits an exact differential formulation on the Liouville torus as well. This integrable system is conjectured to have the Painlevé property with respect to the independent variable u, as in the degenerate cases.

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1. Introduction

We consider the problem, introduced by Ovsiannikov (1956) and by Dyson (1968), of the evolution of a monatomic, isothermal gas cloud of ellipsoidal shape, adiabatically expanding with rotation and precession into a vacuum. Such a motion was known to be describable by a single-particle Hamiltonian, which has been shown (Gaffet 2001) to be Liouville integrable (Landau and Lifshitz 1960, Whittaker 1959) in the absence of vorticity, owing to the presence of two new integrals of motion, denoted as I_6 and L_6, in addition to the integrals of energy (m) and of angular momentum ($\vec{J}^2$, J_z).

The effective dimensionality of the Liouville tori is 3, but that reduces to 2 in the case of motion with an extremal value of the energy, all other integrals of motion being kept fixed and the value of the last integral L_6 is then algebraically related to m, $\vec{J}^2$ and I_6. Gaffet (2003) (hereafter referred to as paper I) has treated the simpler (degenerate) sub-case where the integral L_6 vanishes, which is characterized by the presence on the Liouville torus (Σ) of a single-particle trajectory (L_0) representing a cloud rotating around a fixed axis, whereas all other trajectories exhibit rotation with precession. In those cases, an algebraic transformation was found that changes (Σ) into a quartic surface with eight conic point singularities plus four singularities of a more complex type located on (L_0), which itself has been turned into a straight line by the transformation. (L_0) is a double line on (Σ) and the four point singularities on it have been interpreted (Gaffet 2005, hereafter referred to as paper II) as 'double' conic points, so that (Σ) may be viewed as a degenerate case of a quartic surface with a total of 16 conic points; that view is supported by the fact that, in the general case where L_6 is non-zero, the Liouville torus does present 16 distinct conic point singularities. In paper I, the integration constant Φ along trajectories and the independent variable u (which constitutes the thermasy of the cloud (van Danzig 1939): $u = \int T\,\mathrm{d}t$) were shown to be calculable by quadratures, through the explicit determination of their exact differentials $\mathrm{d}\Phi$, $\mathrm{d}u$, and in paper II the differential system was found to be soluble by separation of variables as well.

In the present work, the restriction of a vanishing L_6 is removed and the line of singularity (L_0) on (Σ) no longer exists. As shown in paper II, the algebraic relation linking the four integrals of motion in the extremal cases admits a rational parametrization in terms of the energy m, total angular momentum $\vec{J}^2$ and of a parameter denoted as K; the numerical calculations on which the present analysis is based concern the values: $m = 5$, $\vec{J}^2 = 12$ and $K = -1$, instead of the value $K = -2$ corresponding to $L_6 = 0$. The above choice is *generic*, and we expect that the methods of resolution developed here, together with the general properties of the solutions, will remain the same for all other non-special values of K (for arbitrarily given m and $\vec{J}^2$).

It was suggested in paper II, based on the results available in the case $L_6 = 0$, that, in the non-degenerate cases where 16 distinct conic points exist on (Σ), i.e. when $L_6 \neq 0$, there might again be a way to group these points into 16 sextuplets, each located on an algebraic surface S_i ($i = 1$–16), and that perhaps there would exist an algebraic transformation that turns each (S_i) into a plane and (Σ) into a quartic surface. In section 4, we show that this is almost, but not completely realized: we do succeed in distributing the points between 16 surfaces (S_i), and in obtaining a coordinate transformation that turns ten of these surfaces into planes and (Σ) into a quartic surface. But in the process one conic point is lost and the quartic (Σ) has only 15 conic points left, instead of the expected 16. The 6th-degree discriminant associated with (Σ) then no longer has a fully separable form and this appears to spoil the separability property of the differential system under study—that is, unless there exists an algebraic transformation turning a quartic with 15 conic point singularities into one with 16.

In sections 2 and 3, we recapitulate the main features of the model of the spinning cloud; and in section 5, we establish the form of the differential system in the new coordinates and show that it is solvable by quadratures; we also obtain explicit expressions of the differentials $\mathrm{d}\Phi$, $\mathrm{d}u$, which are exact differentials defined on the Liouville torus.

2. The model

The ellipsoidal gas clouds considered by Ovsiannikov and by Dyson constitute a Hamiltonian system, which is equivalent to the Hamiltonian governing point-mass motion in a potential in Euclidean nine-dimensional space and time t. When the gas is monatomic (adiabatic index $\gamma = 5/3$), the radial motion separates out and the non-radial motion is governed by a new Hamiltonian on the eight-dimensional unit sphere, with a new canonically conjugate time τ, which is a function of time t. That system is conjectured (and has been shown in various sub-cases) to have the Painlevé property with respect to the independent variable u, distinct from τ, which coincides with the thermasy of the cloud.

The velocity distribution inside the cloud is linear and may be represented by a 3×3 matrix which is a function of time only; it is symmetric, as the flow is assumed vorticity-free, and its traceless part, denoted as v, represents the non-radial part of the motion. The shape of the cloud is described by a diagonal matrix $D = \mathrm{diag}(D_1, D_2, D_3)$ of unit determinant, where $D_{1,2,3}$ are proportional to the principal axes. Its derivative defines the diagonal part of v:

$$\frac{\mathrm{d}}{\mathrm{d}u} \ln D_i = v_{ii} \tag{2.1}$$

whereas the off-diagonal part of v is related to the (instantaneous) angular velocity matrix of the cloud:

$$\omega_{ij} = \frac{D_i^2 + D_j^2}{D_i^2 - D_i^2} v_{ij}. \tag{2.2}$$

The evolution of v is governed by the equation

$$\frac{\mathrm{d}v}{\mathrm{d}u} + v^2 + [v, \omega] - D^{-2} = kI \tag{2.3}$$

meaning that the traceless part of the left-hand side must be zero.

The shape of the cloud may alternatively be represented in a way independent of permutation of the principal axes, through the consideration of the characteristic coefficients X_0, Y_0 of the diagonal matrix D squared:

$$D^6 - X_0 D^4 + Y_0 D^2 - 1 = 0 \tag{2.4}$$

where X_0 is related to the temperature of the cloud and is the potential energy term in the expression of the energy integral m.

The kinetic energy of rotation is: $X_0\, j \cdot \omega$, where $\mathbf{j}$ is the angular momentum vector in the moving frame:

$$j_k = \left(D_i^2 - D_j^2\right) v_{ij} \qquad (i, j, k = \text{circular permutation of } 1, 2, 3). \tag{2.5}$$

Taking account of the remaining kinetic energy term associated with the rate of deformation of the cloud, the energy constant may be written compactly as

$$9m = \left(X_0 X_2 - X_1^2\right) + 3X_0 \tag{2.6}$$

where

$$X_n = \mathrm{Tr}(D v^n D) \qquad (n = 0, 1, 2). \tag{2.7}$$

The characteristic equation of v

$$v^3 + Tv - P = 0 \tag{2.8}$$

provides two additional permutation invariants of interest, T and P; and the simple combination Z_0

$$Z_0 = T - Y_0 \tag{2.9}$$

turns out to play an important role in the formulation of the system.

The system admits two additional integrals, I_6 and L_6, which are polynomials of degree 6 in the components of the velocity matrix v. The leading term (homogeneous and of degree 6) in I_6 is just

$$27P^2 + 4T^3$$

and is the discriminant of the characteristic equation of v. Finally, in terms of the modified angular momentum vector,

$$\tilde{j}_i = -D_i^2 j_i, \tag{2.10}$$

the last integral L_6 assumes the simple form of a triple product

$$L_6 = (\tilde{j}, v\tilde{j}, v^2\tilde{j} - 3j). \tag{2.11}$$

In cases where the cloud rotates about a fixed principal axis, the three vectors in the above product are aligned with the axis and L_6 is necessarily zero; conversely, whenever the constant of motion L_6 is non-zero, the rotation axis must be precessing.

3. The extremal energy cases

In general, for a given set of the five integrals of motion m, I_6, L_6, $\vec{J}^2$, J_z, the phase space reduces to a five-dimensional Liouville torus, but, owing to rotational invariance, two of the coordinates can be ignored (see Landau and Lifshitz 1960, Whittaker 1959) and the torus is effectively three-dimensional only. When the energy of motion reaches its physical minimum value (all other constants of motion being kept fixed), the dimension of the torus shrinks and its effective dimensionality reduces to 2. From an algebraic viewpoint, this reduction in dimension arises from the introduction of a new constraint: that the Jacobian of all the integrals of motion vanishes on the two-dimensional torus.

Although the effective determination of the Jacobian under fully general conditions would be quite difficult in view of the number and of the complicated form of the integrals, we have succeeded in obtaining an explicit relation between integrals of motion:

$$F(m, \alpha^2, \varepsilon, L_6) = 0 \tag{3.1}$$

which expresses the vanishing of the Jacobian (From now on, α^2 denotes the total angular momentum $\vec{J}^2$ and $\varepsilon = -I_6/108$).

When the above constraint is satisfied, the resulting two-dimensional surface (Σ) is found to have 16 conic point singularities, which are the points where the diagonal part of the matrix v vanishes and where, in addition, the determinant of v vanishes as well.

3.1. The extremal energy condition

We are looking for the Liouville tori in which there exists a sub-manifold where the exterior product of the 1-forms $\mathrm{d}m$, $\mathrm{d}\alpha^2$, $\mathrm{d}\varepsilon$, $\mathrm{d}L_6$, vanishes. As, in each Liouville torus, there exists at least one point where the matrix v has the special form

$$v = \begin{pmatrix} 0 & 0 & \beta_2 \\ 0 & 0 & \beta_1 \\ \beta_2 & \beta_1 & 0 \end{pmatrix} \tag{3.2}$$

and the diagonal matrix D with unit determinant is otherwise arbitrary, the four integrals of motion may be viewed as functions in the four-dimensional space $(D_1, D_2, \beta_1, \beta_2)$, and the condition of a vanishing exterior product becomes

$$\frac{\partial(m, \alpha^2, \varepsilon, L_6)}{\partial(D_1, D_2, \beta_1, \beta_2)} = 0. \tag{3.3}$$

This equation has already been solved in the case $L_6 = 0$, through the consideration of the singular solution (denoted (L_0)) which is always present in such cases; and, as it turns out, the generalization to non-zero values of L_6 is straightforward. The four integrals of motion satisfying the vanishing Jacobian condition (5.3) thus depend on three free parameters p, y, K in the following way:

$$\begin{aligned} m &= \frac{(2y+1)(1-K)}{3p} \qquad & \alpha^2 &= \frac{6K(1-y)}{p} \\ \varepsilon &= \frac{2K(x-1)}{x} \qquad & L_6 &= 27\frac{K^2}{xy}(K - 1 + 2y - xy) \end{aligned} \tag{3.4}$$

where $x \equiv p^3/y$.

Whenever m, α^2, ε, L_6 satisfy the above parametric representation, the resulting equation for the unknowns $(D_1, D_2, \beta_1, \beta_2)$ has 16 roots, which correspond to 16 points on the sub-manifold where the Jacobian (5.3) vanishes.

The 16 points are found to satisfy the relations

$$-j \cdot \tilde{j} = c_X X_0 + 3K Z_0 \qquad \tilde{j}^2 = c_X \left(\frac{X_0^2}{1-K} - 2Y_0 \right) - 9K \tag{3.5}$$

where

$$c_X \equiv \frac{3K}{p} \tag{3.6}$$

and in fact, these relations determine a two-dimensional surface (Σ) where the Jacobian vanishes identically and where the 16 points are conic point singularities.

In the same way as for the parametric representation (3.4), equations (3.5) were originally derived as relations satisfied by the singular solution (L_0) which exists in cases $L_6 = 0$, but, as it turns out, they remain meaningful even in the absence of (L_0) and for non-vanishing values of L_6.

In coordinates (X_0, Y_0, Z_0), the equation of the surface reads

$$F(X_0, Y_0, Z_0) = 0 \tag{3.7}$$

where F is a polynomial of degree 10. The surface always intersects itself along a double line (L_2), which lies on a quartic surface $D_4(X_0, Y_0, Z_0) = 0$.

3.2. The degenerate cases

For several special values K_i of the parameter K (the other two free parameters in equation (3.4) being kept fixed), including the value K_0 for which L_6 vanishes, the arrangement of 16 conic point singularities on the Liouville torus collapses into an octet plus a quartet of 'double' conic points which result from the coalescence of four pairs of conic points in the limit $K \to K_i$; and the quartet is located on a double line (L_0) of the surface; these are the degenerate cases.

The key to the integration of the differential system in the degenerate case $L_6 = 0$ is the consideration of the family of surfaces of 7th degree passing through both double lines (L_0) and (L_2); there exists a one-parameter family of such surfaces, depending linearly on a parameter w, and each intersects (Σ) on a second-degree curve (a conic section) in coordinates (X_0, Z_0). The new variable w thus defined is an elliptic function of the independent variable u (the thermasy), and the determination of the remaining variables X_0, Z_0 then depends on the resolution of a Riccati equation, which is of a type integrable by quadratures.

Further, a coordinate transformation has been found which changes (Σ) into a quartic surface and its double line (L_0) into a straight line. There are four more straight lines on the quartic surface, each joining a pair from the octet of conic points and intersecting (L_0). The equation of the surface is of the general form

$$A_2 \rho^2 + B_3 \rho + C_4 = 0 \tag{3.8}$$

where the coefficients are polynomials in the two remaining variables ξ and η, and, through a simple linear transformation, it is amenable to the form of the Stieltjes addition formula (see Goursat 1949) for elliptic functions:

$$\begin{pmatrix} a_4 & a_3/2 & \rho & 1 \\ a_3/2 & a_2 - 2\rho & a_1/2 & -S \\ \rho & a_1/2 & a_0 & P \\ 1 & -S & P & 0 \end{pmatrix} = 0. \tag{3.9}$$

Finally, the differential system is found to be separable in coordinates l_1, l_2, where

$$S = l_1 + l_2 \qquad P = l_1 l_2. \tag{3.10}$$

One of the motivations of the present work was to identify the relevant generalization of the Stieltjes formula, in non-degenerate cases. A natural generalization—which applies to the non-precessing cases with non-extremal values of the energy—involves discriminants of degree 6, instead of the 4th-degree characteristic of elliptic functions. The resulting surface has an equation of a form similar to (3.9) and presents 16 distinct conic points, which can be arranged into 16 sextuplets; each sextuplet is located on a plane (Π_i) $(i = 1, \ldots, 16)$ which remains tangential to the surface along a conic section, and at each conic point six planes intersect.

In the following section, which deals with the non-degenerate cases, we first resolve the problem of identifying the correct distribution of the 16 conic points between 16 sextuplets; and we also identify ten surfaces that play the same role as the planes (Π_i); as a result, a coordinate transformation that turns (Σ) into a quartic surface is found.

4. The reduction of (Σ) to the form of a quartic surface

Our aim here is to find an algebraic transformation that changes the 10th-degree (Σ) into a quartic. In view of the difficulty of the task, our method will be to follow as closely as possible the steps that lead to that result in the degenerate case $L_6 = 0$.

4.1. The eight-dimensional basis of surfaces (S) passing through the double line (L_2)

In the degenerate case (paper II), we were favoured with the occurrence of a variable, denoted as w, which parametrizes surfaces passing through both double lines (L_0) and (L_2), and which turns out to determine plane sections in the new coordinate system where (Σ) is a quartic. This fact led to considerable simplification of the transformation formulae defining the new coordinates in terms of the old, through the use of $w(X_0, Y_0, Z_0)$ as an intermediate variable. Unfortunately, w has no equivalent in non-degenerate cases, so this way of simplifying the form of the transformation is lost. When the expression of w in terms of (X_0, Y_0, Z_0) is substituted, one finds that the *planes* of the new system correspond to 7th-degree surfaces in the old coordinates, passing through the double line (L_2) (since that line is no longer singular in the new coordinates), and in addition possessing the following properties:

(1) The sections $X_0 = \text{constant}$ are of degree 6 only in (Y_0, Z_0).
(2) The degrees in Y_0 and Z_0 are 4 and 5, respectively.
(3) The highest (6th) degree terms in the section have a factor $Y_0 Z_0^4$, i.e., there are only two such terms: $Y_0^2 Z_0^4$ and $Y_0 Z_0^5$.

Thus, each section $X_0 = \text{constant}$ involves 21 *a priori* arbitrary coefficients, which are polynomial functions of X_0. Finally, the degree of the X_0 dependence of the coefficients is constrained so as to minimize the number of independent solutions.

Then, it turns out that there exist just eight linearly independent such polynomials F with the property of identically vanishing along the double line (L_2); the polynomials are arbitrary linear combinations with constant coefficients of a basis of eight particular solutions, which will be denoted as $F_1, \ldots, F_8$. All these results, originally derived in the case $L_6 = 0$, apply to the non-degenerate cases as well.

4.2. The decuplets and the sextuplets of conic singularities

In paper II (see section 5.1 therein), we have shown how to determine the 16 conic points on (Σ) for arbitrary values of the parameter K. We now show how these points can be unambiguously distributed into 16 sextuplets. Our method will be to look for polynomial surfaces (S) of the form

$$F(X_0, Y_0, Z_0) \equiv \lambda_1 F_1 + \cdots + \lambda_8 F_8 = 0 \tag{4.1}$$

passing through a whole sextuplet of conic points. We note that, e.g., λ_8 may be taken to be unity without loss of generality, so that, choosing arbitrarily one sextuplet, there should still be one free parameter left, λ say. From the study of the degenerate case, we hope to find a surface (S) with the additional property that it stays tangential to (Σ) all along its intersection with it, so we fix the value of the remaining free parameter by requiring the surface (S) to be tangential to the tangent cone at one of the conic points. Such a condition of contact assumes the form (Goursat 1949)

$$\begin{pmatrix} \Sigma_{xx} & \Sigma_{yx} & \Sigma_{zx} & F_x \\ \Sigma_{xy} & \Sigma_{yy} & \Sigma_{zy} & F_y \\ \Sigma_{xz} & \Sigma_{yz} & \Sigma_{zz} & F_z \\ F_x & F_y & F_z & 0 \end{pmatrix} = 0 \tag{4.2}$$

where x, y, z mean X_0, Y_0, Z_0, the lower indices indicate partial differentiation and Σ is the 10th-degree polynomial whose vanishing defines the Liouville torus. For such values of λ, in most cases, we find that the surface (S) actually comprises 10 conic points instead of only 6, and that the condition of contact at each point is satisfied. There are 16 such decuplets and each of them determines a complementary sextuplet—since there are 16 conic points in all. On these sextuplets (see the appendix), we find that five members only of the 8-basis $F_1, \ldots, F_8$ are linearly independent, so the condition that (S) passes through the sextuplet involves five independent constraints only instead of six; as a result, there are two free parameters (λ and μ) in that case instead of one, and the condition of contact, at an arbitrarily chosen conic point, merely determines a relation $\mu(\lambda)$. The relation assumes the quadratic form

$$\lambda\mu = a\lambda^2 + b\lambda + c \tag{4.3}$$

and, whenever it holds, the condition of contact is found to be satisfied for the whole sextuplet. As a consequence of equation (4.3), the equation of the resulting one-parameter family of surfaces has the general form

$$F \equiv \lambda^2 A + \lambda B + C = 0 \tag{4.4}$$

where A, B, C are functions of X_0, Y_0, Z_0; their envelope (E) may be found through the elimination of λ between equation (4.4) and the equation

$$F_\lambda \equiv 2\lambda A + B = 0 \tag{4.5}$$

which gives

$$B^2 - 4AC = 0. \tag{4.6}$$

Since A, B, C are 7th-degree polynomials, that is an equation of degree 14, and it is found to have the 10th-degree factor Σ, so that the Liouville torus itself is part of the envelope (E), thus, *the surface* (S) *stays tangential to* (Σ) *all along its intersection with it* and not solely at the conic points.

4.3. The central point of the transformation

We look for surfaces of the form (4.4), that would be changed into planes tangential to (Σ) under some appropriate coordinate transformation, while (Σ) itself would be changed into a quartic surface. As equation (4.4) still involves one free parameter λ, we are now presented with the problem of determining whether all choices of λ may be equivalently good (or bad) or whether some special value(s) should be preferred. In this respect, we remark that if a plane is tangential to a quartic surface all along its intersection curve with it that curve must be a conic, which is a unicursal curve, i.e., one which admits a rational parametrization. Now, all the surfaces of the general form (4.1) have an intersection with (Σ) which is a 12th-degree curve in projection on the plane (X_0, Z_0); and a 6th-degree curve in the case of surfaces of the form (4.4), since they are tangential to (Σ). According to the well-known formula that relates the genus g of a curve of degree N to the number d of its double points

$$g = \frac{(N-1)(N-2)}{2} - d \tag{4.7}$$

for a 6th-degree curve to be unicursal (hence, of zero genus), it must have ten double points. However, we find that for arbitrary values of λ, in general the projection presents less than ten double points, and it turns out that, for the intersection curve to be unicursal, the surface must pass through a point C, the 'central point', which is one of the conic points. Choosing the conic point, the value of the free parameter λ_i ($i = 1, \ldots, 16$) is thereby determined, for each sextuplet.

There are now two essential differences with respect to the degenerate case: in the first place, the 16 surfaces (S_i) no longer all play identical (interchangeable) roles: one can distinguish between the six surfaces associated with a sextuplet that includes point C and the ten that do not. Remarkably, four only of the ten polynomials associated with the latter ten surfaces are linearly independent, so their ratios may be viewed as determining a three-dimensional Cartesian coordinate system, which we shall denote by (X, Y, Z). In that system, the ten surfaces become planes, but the six others do not; and finally, the surface (Σ) itself becomes a quartic surface. The second major difference with respect to the degenerate case is that as the central point always loses its singular nature in this type of transformation (it is turned into a whole (regular) curve on the quartic (Σ)) the conic point C altogether disappears, so that the quartic surface obtained no longer comprises 16 conic points, but only 15.

4.4. The quartic surface

An essential simplifying feature of a quartic surface with conic points is that it admits a parametrization which is rational except for the occurrence of just one square root, through the consideration of rays passing through the conic point.

Choosing then one of the 15 conic points, K_0, as projection point, let us perform the linear transformation (in homogeneous coordinates) that sends K_0 at infinity in the direction of the third coordinate axis, in a new coordinate system denoted as (ξ, η, ρ): the quartic's equation is of the second degree only in ρ, and may be written as

$$A_2\rho^2 + B_3\rho + C_4 = 0 \tag{4.8}$$

where A_2, B_3, C_4 are polynomials in (ξ, η) of degrees 2, 3 and 4, respectively. The coordinates (ξ, η) determine a (vertical) ray through K_0. The discriminant

$$\Delta \equiv B_3^2 - 4A_2C_4 \tag{4.9}$$

vanishes along four straight lines $(\Pi_1), \ldots, (\Pi_4)$ which are the traces of four vertical tangent planes on the horizontal plane (ξ, η) and also vanishes on an ellipse (E_2) which is the trace of a vertical tangent cylinder. That is to say, Δ is decomposable into the product of four linear factors and one quadratic factor:

$$\Delta \equiv \Pi_1 \Pi_2 \Pi_3 \Pi_4 E_2. \tag{4.10}$$

The conic (A_2) which is the locus where $A_2 = 0$, is the trace of the tangent cone at K_0 on the (ξ, η) plane and thus the four lines (Π_i) $(i = 1, \ldots, 4)$ and the conic (E_2) are tangential to it. Being quadratic, (A_2) can be parametrized rationally, by means of a parameter l, say; this opens up the possibility of introducing a modified coordinate system (l_1, l_2) in place of (ξ, η), through the consideration of the two tangents to (A_2) that one can draw through a given point (ξ, η): the new coordinates are then defined to be the values of the parameter l at the two points of contact. Under that transformation, each linear factor Π_i gives rise to a product of the form

$$(l_1 - a_i)(l_2 - a_i) \tag{4.11}$$

a fact which is at the root of the separability property of the degenerate case. In the present case, however, the remaining quadratic factor E_2 no longer decomposes into a product of separable form, even though it does decompose into a product of linear factors:

$$(l_1 + kl_2 - b)(l_2 + kl_1 - b) \tag{4.12}$$

as a consequence of the fact that (E_2) is bi-tangential to (A_2).

Finally, the conic point C, although no longer present as such, still plays an important role in the formulation of the differential system, as we shall see in the next section. That point is stretched under the transformation into a whole curve, whose projection on the (ξ, η) horizontal plane is the quartic:

$$A_4(\xi, \eta) = 0. \tag{4.13}$$

Remarkably, this is a unicursal, having three double points.

5. The differential system and its resolution

We now proceed to reformulate the differential system under study in the framework of the new coordinates (ξ, η). This involves differentiating the transformation formulae, which poses no difficulty since all the functions to be differentiated are polynomials; and it also involves using the inverse transformation formulae.

5.1. Inverse transformation formulae

The direct transformation formulae express the coordinates ξ and η as ratios of seventh-degree polynomials, F_a, F_b, F_d say, in the variables X_0, Y_0, Z_0, of the general form (4.1):

$$\xi = F_a / F_d \qquad \eta = F_b / F_d. \tag{5.1}$$

There are then two possible values for the third variable ρ, depending on the choice of sign for the discriminant Δ. When X_0, Y_0, Z_0 are given, however, there is no sign ambiguity, and ρ is expressed by a formula of the same form as (5.1):

$$\rho = F_c / F_d. \tag{5.2}$$

To determine the form of the inverse formulae, we start with the observation (section 4.3) that surfaces (S) described by equation (4.1) intersect (Σ) on a 12th-degree curve in coordinates X_0, Z_0. In particular, the intersection of a surface $\eta = a\xi + b$ with a section $X_0 =$ constant of (Σ) must consist of 12 points, showing that such a section is a 12th-degree curve in coordinates

(ξ, η) as well. Further, as there exist only two points on the surface (Σ) with given coordinates (ξ, η), the equation of the section must be of degree 2 in X_0, and thus have the general form

$$A_{12}X_0^2 + B_{12}X_0 + C_{12} = 0 \tag{5.3}$$

where A_{12}, B_{12}, C_{12} are polynomials of degree 12 in (ξ, η).

The same reasoning applies to sections $Z_0 =$ constant, which are therefore described by an equation of the same form:

$$A_{12}Z_0^2 + V_{12}Z_0 + W_{12} = 0, \tag{5.4}$$

we note that the leading term coefficient is the same in both cases.

The discriminant of the above second-degree equations (5.3), (5.4) is of course Δ, since, solving for X_0 and Z_0, one must obtain expressions rational in (ξ, η, ρ), as there exists only one point on (Σ) corresponding to that set of coordinates. Thus, the inverse transformation formulae read—using the square root of Δ rather than ρ:

$$X_0 = (-B_{12} + X_9\Delta^{1/2})/2A_{12} \qquad Z_0 = (-V_{12} + Z_9\Delta^{1/2})/2A_{12} \tag{5.5}$$

where X_9, Z_9 are polynomials of degree 9 in (ξ, η). The corresponding inverse transformation formula giving Y_0 is similar, but of higher degree, having the square of A_{12} at its denominator.

5.2. The differential system in the new coordinates (ξ, η)

Given the original differential system and the transformation formulae, direct and inverse, we are now able to determine the derivatives $\xi'(u), \eta'(u)$ corresponding to any given point (ξ, η). As there are in fact two such points, distinguished by the sign of $\sqrt{\Delta}$, we shall denote the corresponding derivatives ξ'^{+}, ξ'^{-} and η'^{+}, η'^{-}. Quantities such as $(\xi'^{+})^2 + (\xi'^{-})^2$ or $\lfloor(\xi'^{+})^2 - (\xi'^{-})^2\rfloor/\sqrt{\Delta}$, together with the corresponding combinations of η derivatives, being uniquely determined, are expected and are found to be rational. Introducing the product (see section 4.4)

$$A_6 \equiv A_2A_4, \tag{5.6}$$

we obtain the following expressions:

$$P_7 = \frac{A_6}{\sqrt{\Delta}}[(\xi'^{+})^2 - (\xi'^{-})^2] \qquad P_{10} = A_6[(\xi'^{+})^2 + (\xi'^{-})^2] \tag{5.7a}$$

$$R_6 = \frac{A_6}{\sqrt{\Delta}}[\xi'^{+}\eta'^{+} - \xi'^{-}\eta'^{-}] \qquad R_9 = A_6[\xi'^{+}\eta'^{+} + \xi'^{-}\eta'^{-}] \tag{5.7b}$$

$$Q_5 = \frac{A_6}{\sqrt{\Delta}}[(\eta'^{+})^2 - (\eta'^{-})^2] \qquad Q_8 = A_6[(\eta'^{+})^2 + (\eta'^{-})^2] \tag{5.7c}$$

and:

$$G_7 = -\sqrt{A_6}\xi'^{+}\xi'^{-} \qquad G_6 = \sqrt{A_6}[\xi'^{+}\eta'^{-} + \xi'^{-}\eta'^{+}] \tag{5.8a}$$

$$G_5 = -\sqrt{A_6}\eta'^{+}\eta'^{-} \qquad G_3 = -\sqrt{A_6}[\xi'^{+}\eta'^{-} - \xi'^{-}\eta'^{+}]/\sqrt{\Delta} \tag{5.8b}$$

where P_{10}, R_9, Q_8 are polynomials in (ξ, η) of degree 10, P_7, R_6, Q_5 are polynomials of degree 7, as well as G_5, G_6, G_7, G_3 is cubic, and the lower index indicates the degree in the variable ξ (the degree in η then follows from the interchangeability of the roles of the two variables: thus, for example, Q_8 is of degree 10 in η, since that is the degree of P_{10} in ξ, R_6 is of degree 6 in both variables, etc).

A number of identities relating these polynomials follow from their above definitions, such as in particular

$$G_6^2 - 4G_5G_7 \equiv G_3^2\Delta. \tag{5.9}$$

We thus obtain expressions of the following form for the quadratic combinations of derivatives:

$$\xi'^2 = [P_{10} + P_7\sqrt{\Delta}]/2A_6 \qquad \xi'\eta' = [R_9 + R_6\sqrt{\Delta}]/2A_6 \qquad \eta'^2 = [Q_8 + Q_5\sqrt{\Delta}]/2A_6 \tag{5.10}$$

(the + signs on ξ' and η' have been omitted).

The derivative with respect to ξ, however,

$$\begin{aligned} p &\equiv \mathrm{d}\eta/\mathrm{d}\xi \\ &\equiv \eta'/\xi', \end{aligned} \tag{5.11}$$

whose consideration leads to the elimination of the variable u, is the root of an equation of second degree only, as may be seen from the identity

$$G_5\xi'^2 + G_6\xi'\eta' + G_7\eta'^2 = 0. \tag{5.12}$$

Taking account of the identity (5.9), one thus obtains the following expression of p:

$$p = -\lfloor G_6 + G_3\sqrt{\Delta}\rfloor/2G_7 \tag{5.13}$$

which constitutes a first-order differential equation (of second degree) for the unknown function $\eta(\xi)$.

5.3. *The integrating factor*

Not unexpectedly, in view of the proven Liouville integrability of the original system, equation (5.13) admits an integrating factor, as we now show. The relation defining the integrating factor may be written in the form

$$\mathrm{d}\Phi = \varphi(\eta'\,\mathrm{d}\xi - \xi'\,\mathrm{d}\eta)/\sqrt{\Delta} \tag{5.14}$$

where $\mathrm{d}\Phi$ is taken to be the exact differential of some unknown function Φ and φ may be termed the integrating factor. In paper II, it was shown that, in the degenerate case ($L_6 = 0$), the integrating factor is unity (Δ being the discriminant associated with the quartic form of the surface (Σ) in all cases).

Let us then consider the 1-form, denoted as $\mathrm{d}\Phi$:

$$\mathrm{d}\Phi = (\eta'\,\mathrm{d}\xi - \xi'\,\mathrm{d}\eta)/\sqrt{\Delta} \tag{5.15}$$

and see whether it does satisfy the condition of integrability of Φ:

$$\mathrm{d} \wedge \mathrm{d}\Phi = 0$$

that is

$$\partial_\xi(\xi'/\sqrt{\Delta}) + \partial_\eta(\eta'/\sqrt{\Delta}) = 0. \tag{5.16}$$

We note that the above condition explicitly involves more than the coefficients of the differential equation (5.12) or (5.13): at least some of the coefficients of equations (5.10) also come into play. We shall try to eliminate them, so as to obtain an integrability condition that involves only Δ, A_6 and the coefficients G.

First, noting that (3.17) really represents two distinct equations

$$\partial_\xi(\xi'^+/\sqrt{\Delta}) + \partial_\eta(\eta'^+/\sqrt{\Delta}) = 0 \qquad \partial_\xi(\xi'^-/\sqrt{\Delta}) + \partial_\eta(\eta'^-/\sqrt{\Delta}) = 0, \tag{5.17}$$

we obtain, forming appropriate combinations of this pair,

$$\pi\delta\partial_\eta \ln(\xi'^-/\xi'^+) = 2\pi_\xi + \sigma\pi_\eta + 2\pi\sigma_\eta \tag{5.18}$$

where σ, π, δ are defined by

$$\sigma \equiv p^+ + p^- = -G_6/G_7 \qquad \delta \equiv p^+ - p^- = -\sqrt{\Delta}G_3/G_7 \tag{5.19a}$$

and

$$\pi \equiv \xi'^+\xi'^-/\Delta = G_7\big/\big(\Delta A_6^{1/2}\big). \tag{5.19b}$$

Exchanging the roles of ξ, η, one obtains another equation symmetrical to (5.18):

$$\tilde{\pi}\tilde{\delta}\partial_\xi \ln(\eta'^-/\eta'^+) = 2\tilde{\pi}_\xi + \tilde{\sigma}\tilde{\pi}_\xi + 2\tilde{\pi}\tilde{\sigma}_\xi \tag{5.20}$$

where $\tilde{\sigma}, \tilde{\delta}, \tilde{\pi}$, as well as σ, δ, π, merely involve the coefficients G, Δ and A_6.

To complete the elimination of the polynomials P, Q, R occurring in equations (5.10), we note that the ratios ξ'^-/ξ'^+, η'^-/η'^+, which occur in equations (5.18), (5.20) through their logarithmic derivatives, may be written as

$$\xi'^-/\xi'^+ = h\sqrt{G} \qquad \eta'^-/\eta'^+ = h/\sqrt{G} \tag{5.21}$$

where

$$G \equiv (G_6 + G_3\sqrt{\Delta})/(G_6 - G_3\sqrt{\Delta}). \tag{5.22}$$

Thus, equations (5.18), (5.20) determine both partial derivatives of $\ln h$. The resulting condition of integrability is

$$\begin{aligned}
&2G_7\partial_{\xi\xi}\ln\big(\Delta A_6^{1/2}\big/G_7\big) - 2G_6\partial_{\xi\eta}\ln\big(\Delta A_6^{1/2}\big/G_6\big) + 2G_5\partial_{\eta\eta}\ln\big(\Delta A_6^{1/2}\big/G_5\big)\\
&\quad - 8\frac{G_5G_6G_7}{G_3^2\Delta}\partial_\eta \ln\frac{(G_5G_7)^{1/2}}{G_6}\partial_\xi \ln\frac{(G_5G_7)^{1/2}}{G_6}\\
&\quad + \partial_\xi \ln\big(\Delta A_6^{1/2}\big/G_7\big)\lfloor G_6\partial_\eta \ln(G_3\sqrt{\Delta}/G_6) - 2G_7\partial_\xi \ln(G_3\sqrt{\Delta}/G_7)\rfloor\\
&\quad + \partial_\eta \ln\big(\Delta A_6^{1/2}\big/G_5\big)\lfloor G_6\partial_\xi \ln(G_3\sqrt{\Delta}/G_6) - 2G_5\partial_\eta \ln(G_3\sqrt{\Delta}/G_5)\rfloor = 0.
\end{aligned} \tag{5.23}$$

The left-hand side of the above equation is a polynomial, as the residues are all found to vanish identically whenever the identity (5.9) is taken into account. Substituting the polynomials $A_6, \Delta, G_3, G_5, G_6, G_7$ obtained in section 5.2 (see table 1), the equation is found to be identically satisfied, thus showing that the proposed integrating factor is indeed correct.

5.4. Exact differential expression of the independent variable u

The independent variable u, which represents the thermasy of the cloud, can be determined by quadratures, in the same way as the integration constant Φ—thus completing the integration of the differential system—through the explicit determination of its exact differential over the Liouville torus.

First, we remark that the second equation (5.17) expresses integrability of $\tilde{\Phi}$, defined by (see equation (5.15))

$$\begin{aligned}
\mathrm{d}\tilde{\Phi} &= (\xi'^-\,\mathrm{d}\eta - \eta'^-\,\mathrm{d}\xi)/\sqrt{\Delta}\\
&= \left(\frac{G_5}{\eta'}\,\mathrm{d}\xi - \frac{G_7}{\xi'}\,\mathrm{d}\eta\right)\Bigg/(\Delta A_6)^{1/2}.
\end{aligned} \tag{5.24}$$

Unlike Φ itself, $\tilde{\Phi}$ does not stay constant along trajectories:

$$\frac{\mathrm{d}\tilde{\Phi}}{\mathrm{d}u} = G_3\big/A_6^{1/2} \tag{5.25}$$

Table 1. We list here, for the values $m = 5$, $\vec{J}^2 = 12$ and $K = -1$ of the constants of motion, the coefficients of the polynomials A_6 and Δ in the variables ξ, η (third and fourth columns, respectively). The first two columns indicate the exponents, in ξ and η respectively, of each term.

		A_6	Δ
		Degree 6	
0	6	−0.726 0651	1
1	5	−0.851 1177	2.764 143
2	4	0.216 4935	3.548 825
3	3	0.584 0778	3.095 538
4	2	0.205 4162	1.720 218
5	1	$9.545\,3607 \times 10^{-3}$	0.449 5617
6	0	$-3.185\,1367 \times 10^{-3}$	$4.184\,7281 \times 10^{-2}$
		Degree 5	
0	5	1.494 538	0.125 0557
1	4	2.838 860	0.142 7436
2	3	1.602 926	0.428 4779
3	2	0.140 0996	0.378 5971
4	1	$-9.809\,9336 \times 10^{-2}$	$-3.539\,8189 \times 10^{-2}$
5	0	$-1.787\,2103 \times 10^{-2}$	$-2.546\,5362 \times 10^{-2}$
		Degree 4	
0	4	−0.574 5899	0.303 0852
1	3	−1.502 971	0.284 5955
2	2	−1.092 202	0.444 5097
3	1	−0.231 3061	0.173 4675
4	0	$-5.464\,1678 \times 10^{-3}$	$9.008\,8621 \times 10^{-2}$
		Degree 3	
0	3	$-9.784\,3543 \times 10^{-2}$	−0.151 8164
1	2	0.155 9026	0.116 7452
2	1	0.235 1518	$5.063\,8434 \times 10^{-2}$
3	0	$4.918\,6110 \times 10^{-2}$	$9.784\,2023 \times 10^{-2}$
		Degree 2	
0	2	0.102 1210	$6.458\,9255 \times 10^{-2}$
1	1	$7.632\,4008 \times 10^{-2}$	$-2.238\,5262 \times 10^{-2}$
2	0	$-6.338\,9530 \times 10^{-3}$	$7.892\,0856 \times 10^{-2}$
		Degree 1	
0	1	$-3.060\,0622 \times 10^{-2}$	$-4.467\,8845 \times 10^{-3}$
1	0	$-2.158\,3933 \times 10^{-2}$	$1.967\,2729 \times 10^{-2}$
		Degree 0	
0	0	$4.576\,0009 \times 10^{-3}$	$-8.992\,8415 \times 10^{-3}$

so, if $\mathrm{d}u$ is an exact differential which coincides with the differential of u along trajectories, it must have the general form

$$\mathrm{d}u = (\sqrt{A_6}\,\mathrm{d}\tilde{\Phi} - P\,\mathrm{d}\Phi)/G_3 \tag{5.26}$$

where P is some coefficient, a function of position on (Σ). The integrability condition

$$\mathrm{d} \wedge \mathrm{d}u = 0$$

gives rise to the following condition on P:

$$\frac{\mathrm{d}}{\mathrm{d}u}(P/G_3) = \frac{1}{G_3}\left[\frac{G_7}{\xi'}\partial_\xi \ln\left(A_6^{1/2}/G_3\right) + \frac{G_5}{\eta'}\partial_\eta \ln\left(A_6^{1/2}/G_3\right)\right]. \tag{5.27}$$

This is a first-order partial differential equation on P and it is found to admit a solution of the form

$$P = P_3 + k_0\sqrt{\Delta} \tag{5.28}$$

where P_3 is polynomial and k_0 is a constant. Using the above ansatz (5.28) for P, leads to a pair of ordinary differential equations for P_3, of the form

$$G_3\partial_\xi P_3 - P_3\partial_\xi G_3 = U_5 \qquad G_3\partial_\eta P_3 - P_3\partial_\eta G_3 = V_5 \tag{5.29}$$

where U_5 and V_5 are rational fractions which become polynomials (of degree 5) for some particular value of k_0; and as P_3 is assumed to be polynomial, the value of the constant k_0 is thus determined. Then the two equations are found to be compatible, P_3 can be obtained by quadratures, is found to be free of logarithmic terms, and turns out to be a cubic polynomial in (ξ, η), which is determined up to an arbitrary additive term proportional to G_3.

The introduction of k_0 brings about a notable simplification in the formulation of the differential system: the seventh-degree polynomials P_7, R_6, Q_5 defined in section 5.2 are found to be expressible in terms of polynomials X_4, Y_3, Z_2 of the fourth degree only in ξ, η

$$P_7 \equiv G_3X_4 - 2k_0G_7 \qquad R_6 \equiv G_3Y_3 + k_0G_6 \qquad Q_5 \equiv G_3Z_2 - 2k_0G_5 \tag{5.30}$$

and then, making use of the identities,

$$-G_3P_{10} \equiv G_6P_7 + 2G_7R_6 \quad G_3R_9 \equiv G_5P_7 - G_7Q_5 \quad G_3Q_8 \equiv G_6Q_5 + 2G_5R_6 \tag{5.31}$$

which result from the definitions (5.7) and (5.8), the 10th-degree polynomials P_{10}, R_9, Q_8 admit an explicitly polynomial expression as well:

$$P_{10} \equiv -(G_6X_4 + 2G_7Y_3) \qquad R_9 \equiv (G_5X_4 - G_7Z_2) \qquad Q_8 \equiv (G_6Z_2 + 2G_5Y_3). \tag{5.32}$$

The new polynomials satisfy a relation

$$X_4\eta'^2 - 2Y_3\eta'\xi' + Z_2\xi'^2 = -G_3\sqrt{\Delta} \tag{5.33}$$

and also satisfy the pair of identities

$$G_5X_4 + G_6Y_3 + G_7Z_2 \equiv -k_0G_3\Delta \tag{5.34}$$

$$Y_3^2 - X_4Z_2 \equiv k_0^2\Delta + A_6, \tag{5.35}$$

hence, whenever the discriminant vanishes:

$$X_4p^2 - 2Y_3p + Z_2 = 0. \tag{5.36}$$

At conic points, which are the double points of the locus $\Delta = 0$, the above equation must be satisfied for the two distinct corresponding values of the slope $p \equiv \mathrm{d}\eta/\mathrm{d}\xi = -\Delta_\xi/\Delta_\eta$; together with equation (5.35) that constitutes a complete system of equations from which the values of X_4, Y_3, Z_2 can be deduced. As there are five conic points (in addition to the projection point, K_0) on each line $\Pi_1, \ldots, \Pi_4$, their values on these lines can be determined as well; finally, as there are four lines and Y_3, Z_2 are at most cubic in ξ, their values can be determined everywhere, through Lagrange interpolation; similarly, since X_4 is only quadratic in η, it can be obtained too.

Then the identities (5.9) and (5.34) are found to be sufficient for the determination of the polynomials $G_{5,6,7}$ and G_3, except that one finds a whole one-parameter family of solutions, quadratically depending on a parameter z. Finally, the cubic polynomial P_3 which occurs in the expression of the differential du is found to be linearly dependent on z; it is worth noting that the indeterminacy of P_3 modulo G_3 is thus removed, the latter being quadratic in z.

In this way, all the relevant polynomials can be determined, given only the discriminant Δ, A_6 and the free parameter z.

6. Conclusion

We have presented an explicit resolution of the differential system governing the evolution of a spinning gas cloud with precessing motion around its instantaneous axis of rotation, in cases where the integral of motion L_6 is non-zero. Although some of the numerical calculations on which the present analysis is based have been done assuming a definite value of that constant, the algebraic form of the surface (Σ) (the Liouville torus) indicates that this value is generic, so that the results of the present work should be applicable to all non-special values of L_6. This constitutes a one-parameter extension of the domain of known solutions with precession.

One essential result is that, as in the cases where $L_6 = 0$, the Liouville torus can be algebraically transformed into a surface of the fourth degree, having several conic point singularities; but that surface is 'non-degenerate', in that it does not have a double line of self-intersection, where eight of the conic points coalesce into a quartet of 'double conic points'. Another essential difference is that there are only 15 conic points left on the surface instead of 16, as one of the 16 points present on the original surface has been destroyed by the transformation. As a result, the property of the system of being soluble by separation of variables appears to be lost (unless there exists an algebraic transformation that changes a quartic surface with 15 conic points into one with 16). The solution of the system no longer involves elliptic functions, but it still admits an integrating factor (section 5.3), and the independent variable, u, similarly admits an exact differential formulation (section 5.4) on the Liouville torus.

Appendix. The 16 sextuplets of conic points

We give here, as an example, the equation for the conic points in the case $K = -1$ (which corresponds to the constant of motion $L_6 = 120$) and the distribution of the points between sextuplets, which results from the analysis of section 4.2. The points are associated with the roots of the equation

$$\begin{aligned} P(D) \equiv {} & 20\,736D^{16} - 262\,656D^{15} + 525\,312D^{14} + 3798\,528D^{13} - 5406\,208D^{12} \\ & + 4793\,472D^{11} - 31\,897\,344D^{10} + 31\,055\,360D^{9} - 10\,650\,528D^{8} \\ & + 7471\,008D^{7} + 5404\,608D^{6} - 452\,736D^{5} + 1801\,776D^{4} \\ & - 347\,144D^{3} + 150\,768D^{2} - 32\,712D + 5425 = 0 \end{aligned} \tag{A.1}$$

where D is a parameter related to the (ellipsoidal) shape of the cloud, in terms of which explicit expressions of the coordinates (X_0, Y_0, Z_0), valid at conic points, are available. We list below the roots D (with a few significant figures only, for compactness) and their arrangement into 16 sextuplets Π:

$$\begin{gathered} D_1 = -3.66 \qquad D_2 = -0.55 \qquad D_3 = 1.15 \qquad D_4 = 1.90 \\ D_5 = (-0.68, 1.66) \qquad D_6 = (-0.08, 0.31) \qquad D_7 = (-0.01, 0.41) \\ D_8 = (0.15, 0.13) \qquad D_9 = (0.19, 0.59) \qquad D_{10} = (7.34, 2.33) \\ D_{11} = D_5^*, \ldots, D_{16} = D_{10}^*. \end{gathered}$$

Denoting 1, 2, . . . , 16 the corresponding conic points, the sextuplets are the following:

$$
\begin{array}{ll}
\Pi_1 : 1\ \ 4\ \ 5\ \ 10\ \ 11\ \ 16 & \Pi_2 : 1\ \ 4\ \ 8\ \ 9\ \ 14\ \ 15 \\
\Pi_3 : 2\ \ 3\ \ 5\ \ 8\ \ 11\ \ 14 & \Pi_4 : 2\ \ 3\ \ 9\ \ 10\ \ 15\ \ 16 \\
\Pi_5 : 1\ \ 3\ \ 6\ \ 8\ \ 10\ \ 13 & \Pi_6 : 5\ \ 7\ \ 8\ \ 13\ \ 15\ \ 16 \\
\Pi_7 : 3\ \ 4\ \ 6\ \ 7\ \ 11\ \ 15 & \Pi_8 : 5\ \ 6\ \ 10\ \ 12\ \ 14\ \ 15 \\
\Pi_9 : 1\ \ 2\ \ 11\ \ 12\ \ 13\ \ 15 & \Pi_{10} : 2\ \ 4\ \ 6\ \ 13\ \ 14\ \ 16 \\
\Pi_{11} : 1\ \ 3\ \ 7\ \ 12\ \ 14\ \ 16 & \Pi_{12} : 7\ \ 9\ \ 10\ \ 11\ \ 13\ \ 14 \\
\Pi_{13} : 3\ \ 4\ \ 5\ \ 9\ \ 12\ \ 13 & \Pi_{14} : 6\ \ 8\ \ 9\ \ 11\ \ 12\ \ 16 \\
\Pi_{15} : 1\ \ 2\ \ 5\ \ 6\ \ 7\ \ 9 & \Pi_{16} : 2\ \ 4\ \ 7\ \ 8\ \ 10\ \ 12.
\end{array}
$$

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